

# TRANSFER OPERATOR FOR THE GAUSS' CONTINUED FRACTION MAP. I. STRUCTURE OF THE EIGENVALUES AND TRACE FORMULAS

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**ABSTRACT.** Let  $\mathcal{L}$  be the transfer operator associated with the Gauss' continued fraction map, known also as the *Gauss-Kuzmin-Wirsing operator*, acting on the Banach space. In this work we prove an asymptotic formula for the eigenvalues of  $\mathcal{L}$ . This settles, in a stronger form, the conjectures of D. Mayer and G. Roepstorff (1988), A.J. MacLeod (1992), Ph. Flajolet and B. Vallée (1995), also supported by several other authors. Further, we find an exact series for the eigenvalues, which also gives the canonical decomposition of trace formulas due to D. Mayer (1976) and K.I. Babenko (1978). This crystallizes the contribution of each individual eigenvalue in the trace formulas.

## 1. CONJECTURES AND THE MAIN RESULT

Our first attack of this problem, based on the techniques developed in [1], is presented in [2]. Though the obtained result is highly supported by numerical computations, the series for the eigenvalues in [2] is too complex and does not give neither the structure, nor asymptotics. Here we present another approach which does answer these questions. In the second part of this paper we explicitly compute the first constants in the asymptotic expansion and investigate a refined arithmetic structure of functions  $W_\ell(\mathbf{X})$ .

**1.1. Introduction.** Let  $\mathbb{D}$  be the disc  $\{x \in \mathbb{C} : |z - 1| < \frac{3}{2}\}$ . Let  $\mathbf{V}$  be the Banach space of functions which are analytic in  $\mathbb{D}$  and are continuous in its closure, with the supremum norm. *The Perron-Frobenius, or the transfer operator* for the Gauss' continued fraction map, also called *the Gauss-Kuzmin-Wirsing operator*, is defined for functions  $f \in \mathbf{V}$  by [11, 12, 13, 18, 24]

$$\mathcal{L}[f(t)](z) = \sum_{m=1}^{\infty} \frac{1}{(z+m)^2} f\left(\frac{1}{z+m}\right). \quad (1)$$

Our chief interest is the point spectrum of this operator. As was shown in [3, 4], this operator is of trace class and is nuclear of order 0. Thus, it possesses the eigenvalues  $\lambda_n$ ,  $n \in \mathbb{N}$ , which are real numbers,  $|\lambda_n| \geq |\lambda_{n+1}|$ ,  $\lambda_1 = 1$ , and  $\sum_{n=1}^{\infty} |\lambda_n|^\epsilon < +\infty$  for every  $\epsilon > 0$ . In fact, the Hilbert-Schmidt operator  $\mathcal{K}$ , defined by

$$\mathcal{K}[u(t)](x) = \int_0^{\infty} \frac{J_1(2\sqrt{xy})}{\sqrt{(e^x - 1)(e^y - 1)}} u(y) dy,$$

for  $u$  belonging to the Hilbert space  $L^2(\mathbb{R}_+, m)$ ,  $dm(y) = \frac{y}{e^y - 1} dy$ , has the same point spectrum, counting algebraic multiplicities [3, 17, 18]. The transfer operator  $\mathcal{L}$  arises from the Gauss map  $F(x) =$

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$\{1/x\}$ ,  $x \in (0, 1]$ ,  $F(0) = 0$ . Let  $F^{(1)} = F$ , and  $F^{(k)} = F \circ F^{(k-1)}$  for  $k \geq 2$ . As is now well-known, due to important contributions by Gauss, Kuzmin, Lévy, Wirsing, Babenko, Babenko and Jur'ev, Mayer, we have

$$\mu(a \in [0, 1] : F^{(k)}(a) < z) = \frac{\log(1+z)}{\log 2} + \sum_{n=2}^{\infty} \lambda_n^k \Phi_n(z).$$

Here  $\mu(\star)$  stands for the Lebesgue measure, and for each  $n \geq 2$ , the eigenfunction  $\Phi_n(z)$  is defined in the cut plane  $\mathbb{C} \setminus (-\infty, -1]$ , it satisfies the boundary conditions  $\Phi_n(0) = \Phi_n(1) = 0$ , the regularity condition

$$\sup_{\Re(z) \geq -\frac{1}{2}} |(z+1)U(z)| < +\infty. \quad (2)$$

where  $U(z) = \Phi'_n(z)$ , and the functional equation

$$\Phi_n(z+1) - \Phi_n(z) = \frac{1}{\lambda_n} \cdot \Phi_n\left(\frac{1}{z+1}\right).$$

Thus,  $\Phi_1(z) = \frac{\log(1+z)}{\log 2}$ . The eigenfunctions of  $\mathcal{L}$  are then given by  $\Phi'_n(z)$ ,  $n \in \mathbb{N}$ . Moreover, every function  $U(z)$  which satisfies, for a certain  $\lambda \in \mathbb{R} \setminus \{0\}$ , the functional equation

$$U(z) = U(z+1) + \frac{1}{\lambda(z+1)^2} U\left(\frac{1}{z+1}\right), \quad z \in \mathbb{C} \setminus (-\infty, -1], \quad (3)$$

and the regularity property (2), is the egenfunction of  $\mathcal{L}$  with the eigenvalue  $\lambda$ . More details can be found in [3, 13, 24].

The nature of the eigenvalues  $\lambda_n$  for  $n \geq 2$  is unknown. It is widely believed that these constants are unrelated to other most important constants in mathematics; in particular, it is expected that they are neither algebraic numbers nor periods. Now, more that 480 digits of  $\lambda_2$  have been calculated [5], but one can get rigorous certificates only for the several first digits of  $\lambda_2$  and  $\lambda_3$  [8, 13, 15, 17, 25]. On the other hand, the trace of the operator  $\mathcal{L}$  can be given explicitly. As was shown in [16] (see also [6, 7, 14, 17, 18]), we have

$$\begin{aligned} \text{Tr}(\mathcal{L}) &= \sum_{n=1}^{\infty} \lambda_n = \int_0^{\infty} \frac{J_1(2x)}{e^x - 1} dx = \sum_{\ell=1}^{\infty} \frac{1}{\xi_{\ell}^{-2} + 1} = \frac{1}{2} - \frac{1}{2\sqrt{5}} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \binom{2k}{k} (\zeta(2k) - 1), \\ \text{Tr}(\mathcal{L}^2) &= \sum_{n=1}^{\infty} \lambda_n^2 = \int_0^{\infty} \int_0^{\infty} \frac{J_1(2\sqrt{xy})^2}{(e^x - 1)(e^y - 1)} dx dy = \sum_{i,j=1}^{\infty} \frac{1}{(\xi_{i,j} \xi_{j,i})^{-2} - 1}; \end{aligned}$$

here

$$\xi_{\ell} = \frac{1}{\ell+} \frac{1}{\ell+} \frac{1}{\ell+} \cdots, \quad \xi_{i,j} = \frac{1}{i+} \frac{1}{j+} \frac{1}{i+} \frac{1}{j+} \cdots, \quad \ell, i, j \in \mathbb{N}.$$

Without going into detail, we note that these trace formulas and this field is deeply and intricately related to the Selberg zeta function, the Riemann zeta function, Maass wave forms and modular forms for the full modular group [14, 19].

Throughout this paper, we fix the notation

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

The constants  $\lambda_n$  have received a considerable amount of attention in recent decades. Nevertheless, there were three outstanding unresolved problems; no theoretical progress was made towards any of them. As was said before, we henceforth arrange the eigenvalues according to their absolute value  $|\lambda_1| \geq |\lambda_2| \geq \dots$ . Of course, in case  $\lambda_n = \pm \lambda_{n+1}$  for some  $n$ , this arrangement is not uniquely defined. Despite this, we have

**Conjecture.** *The following three statements are true:*

- i) *Simplicity. The eigenvalues are simple,  $|\lambda_n|$  strictly decreases.*
- ii) *Sign. The eigenvalues have alternating sign:  $(-1)^{n+1}\lambda_n > 0$ .*
- iii) *Ratio. There exists  $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_{n+1}} = -\frac{3+\sqrt{5}}{2} = -\phi^2$ .*

The first and the second conjectures can be attributed to Mayer and Roepstorff [18], also reiterated by MacLeod, and the last was raised by MacLeod [15] (only with a constant  $\approx -2.6$ ), and seconded by Flajolet and Vallée [6, 8, 9, 10] (with the constant  $-\phi^2$ ). Several authors also claimed to believe these conjectures. The ratio conjecture has the following explanation. The spectrum of the operator

$$\mathcal{L}_0[f(t)](x) = \frac{1}{(x+1)^2} f\left(\frac{1}{x+1}\right), \quad f \in \mathbf{V},$$

is given by  $(-1)^{n+1}\phi^{-2n}$ ,  $n \in \mathbb{N}$ . It is expected that the terms in (1) for  $m \geq 2$  act only as small perturbations to  $\mathcal{L}_0$ . Of course, the “Sign” and “Simplicity” conjectures follow from the “Ratio” conjecture for sufficiently large  $n$  (provided it is effective and we can verify these conjectures for the first values of  $n$ ).

**1.2. Main results.** It is surprising that in fact the real asymptotics of the sequence  $\lambda_n$ , minding the above remark about  $\mathcal{L}_0$ , is as simple as is allowed.

**Theorem 1** (Asymptotics). *We have the formula*

$$(-1)^{n+1}\lambda_n = \phi^{-2n} + c(n) \cdot \frac{\phi^{-2n}}{\sqrt{n}},$$

where  $0.4 < c(n) < 1.7$ , and  $c(n)$  tends to a limit, as  $n \rightarrow \infty$ . Further, there exists an asymptotic expansion of the form

$$(-1)^{n+1}\lambda_n \sim \phi^{-2n} + \phi^{-2n} \sum_{p=1}^{\infty} d(p)n^{-p/2}, \quad d(p) \in \mathbb{R}.$$

Based on the computation of P. Sebah [22], we have:

$$\begin{aligned} c(1) &= 1.618_+, & c(2) &= 1.529_+, & c(3) &= 1.403_+, & c(10) &= 1.223_+, & c(20) &= 1.184_+, \\ c(50) &= 1.153_+, & c(70) &= 1.145_+, & c(100) &= 1.137_+, & c(149) &= 1.1313_+, & c(150) &= 1.1312_+. \end{aligned}$$

Our second result gives the structure of the eigenvalues. Let  $P_m^{(\alpha, \beta)}(x)$  stand for the classical Jacobi polynomials [23].

**Theorem 2** (Arithmetic and decomposition of trace formulas). *There exist functions  $W_v(\mathbf{X})$ ,  $v \geq 0$ , defined by  $W_0(\mathbf{X}) = 1$ ,  $W_1(\mathbf{X}) = \frac{5}{4} \cdot \phi^{-2\mathbf{X}} P_{\mathbf{X}-1}^{(0,1)}(3/2)$ , and then by a certain explicit recurrence - this will be given later, see (7) - such that*

$$(-1)^{n+1} \lambda_n = \phi^{-2n} \sum_{\ell=0}^{\infty} W_{\ell}(n), \quad |W_{\ell}(n)| < \frac{C}{\ell^2 \sqrt{n}} \text{ for } \ell \geq 1,$$

for an absolute constant  $C$ . This decomposition is compatible and gives the decomposition of trace formulas for the powers of  $\mathcal{L}$ : for the first and second powers, we have, respectively,

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-2n} W_{\ell-1}(n) &= \frac{1}{\xi_{\ell}^{-2} + 1}, \quad \ell \geq 1, \\ \sum_{i+j=\ell} \sum_{n=1}^{\infty} \phi^{-4n} W_{i-1}(n) W_{j-1}(n) &= \sum_{i+j=\ell} \frac{1}{(\xi_{i,j} \xi_{j,i})^{-2} - 1}, \quad \ell \geq 2. \end{aligned}$$

Analogously for higher powers of  $\mathcal{L}$ .

So, in the trace formulas now we are able to crystallize the contribution of each individual eigenvalue. Thus, this defines an infinite matrix whose elements in rows add up to eigenvalues, and elements in columns add up to  $(\xi_{\ell}^{-2} + 1)^{-1}$ . The last two equalities can be checked to hold true for, say,  $\ell = 1, 2$  (in the first case) and  $\ell = 3$  (in the second), if we use the representation of Jacobi polynomials in terms of integral (6):

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-2n} &= \frac{1}{\phi^2 + 1} = \frac{1}{\xi_1^{-2} + 1}, \\ \frac{5}{4} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-4n} P_{n-1}^{(0,1)}(3/2) &= \frac{1}{2\sqrt{2} + 4} = \frac{1}{\xi_2^{-2} + 1}, \\ \frac{5}{4} \sum_{n=1}^{\infty} \phi^{-6n} P_{n-1}^{(0,1)}(3/2) &= \frac{1}{4\sqrt{3} + 6} = \frac{1}{(\xi_{1,2} \xi_{2,1})^{-2} - 1}. \end{aligned}$$

MAPLE re-confirms this, too. Further, the last theorem, for the first time, gives rigorous certificates (though considerable computational time and space resources are needed) to calculate numerically the first few digits of an eigenvalue with any index.

**Corollary.** *All three claims of the Conjecture are true.*

## 2. PRELIMINARIES

**2.1. Jacobi polynomials.** For  $\alpha, \beta \in \mathbb{Z}$ ,  $m \in \mathbb{N}_0$ , the classical Jacobi polynomials are given by [23]

$$(x-1)^{\alpha} (1+x)^{\beta} P_m^{(\alpha, \beta)}(x) = \frac{1}{2^{m+1} \pi i} \oint \frac{(w-1)^{m+\alpha} (w+1)^{m+\beta}}{(w-x)^{m+1}} dw; \quad (4)$$

here a small contour winds  $w = x$  in the positive direction. For arbitrary  $x$  outside the closed interval  $[-1, 1]$ , one has an asymptotic formula

$$P_m^{(\alpha, \beta)}(x) \sim \frac{((x+1)^{1/2} + (x-1)^{1/2})^{\alpha+\beta}}{(x-1)^{\alpha/2} (x+1)^{\beta/2} (x^2-1)^{1/4}} \cdot \frac{1}{\sqrt{2\pi m}} \cdot \left(x + (x^2-1)^{1/2}\right)^{m+1/2}, \quad \text{as } m \rightarrow \infty.$$

We will now need the case  $m = \mathbf{X} - 1$ ,  $(\alpha, \beta) = (0, 1)$  (see Theorem 2). Moreover, one can extract the second asymptotic term. In this particular case this reads as

$$\frac{5}{4}P_{\mathbf{X}-1}^{(0,1)}(3/2) \sim \frac{\phi^{2\mathbf{X}}}{\sqrt{\mathbf{X}}} \cdot \frac{5^{1/4}}{2\sqrt{\pi}} + O\left(\frac{\phi^{2\mathbf{X}}}{\mathbf{X}}\right).$$

(Here and in the sequel one can think of  $\mathbf{X} = n$ , exactly the index of an eigenvalue  $\lambda_n$ , though we choose to use an unspecified variable to denote that it is a function in  $\mathbf{X}$ ).

**2.2. g-coefficients of the analytic function.** We will now introduce some special coefficients.

**Proposition 1.** *Every analytic in the half-plane  $\Re(z) > -\frac{1}{2}$  function  $f(z)$  can be expanded in the following way:*

$$f(z) = \sum_{j=1}^{\infty} a_j \frac{(z - \phi^{-1})^{j-1}}{(z + \phi)^{j+1}},$$

where  $|a_j| < C(f, \epsilon) \cdot (1 + \epsilon)^j$  for every  $\epsilon > 0$ . We call  $a_j$  the  $j$ th golden coefficient, or  $g$ -coefficient, of the analytic function  $f(z)$ .

*Proof.* Note a simple identity

$$\oint_{\mathcal{C}} \frac{(z - \phi^{-1})^{\ell-1}}{(z + \phi)^{\ell+1}} \cdot \frac{(z + \phi)^j}{(z - \phi^{-1})^j} dz = \begin{cases} \frac{2\pi i}{2\phi-1}, & \text{if } j = \ell, \\ 0, & \text{if } j \neq \ell; \end{cases} \quad (5)$$

here a small contour  $\mathcal{C}$  winds the point  $\phi^{-1}$  in the positive direction. So, we define  $j$ th  $g$ -coefficient of the function  $f(z)$  by the formula

$$a_j = \frac{2\phi - 1}{2\pi i} \oint_{\mathcal{C}} f(z) \cdot \frac{(z + \phi)^j}{(z - \phi^{-1})^j} dz.$$

Let

$$w = \frac{z - \phi^{-1}}{z + \phi}, \quad z = \frac{w\phi + \phi^{-1}}{-w + 1}.$$

The  $z$ -half plane  $\Re(z) > -\frac{1}{2}$  is mapped in the  $w$ -plane to the disc  $|w| < 1$ . Given  $f$ , analytic in  $\Re(z) > -\frac{1}{2}$ . Let

$$g(w) = (z + \phi)^2 f(z) = \left(\frac{2\phi - 1}{w - 1}\right)^2 f\left(\frac{w\phi + \phi^{-1}}{-w + 1}\right).$$

Then  $g(w)$  is analytic inside  $|w| < 1$  and  $a_j$  is its Taylor coefficient at  $w^{j-1}$ . □

For the example, the  $g$ -coefficients of the dominant eigenfunction of  $\mathcal{L}$ , namely,  $f(z) = \frac{1}{z+1}$ , are given by

$$a_j = \frac{2\phi - 1}{2\pi i} \oint_{\mathcal{C}} \frac{1}{z + 1} \cdot \frac{(z + \phi)^j}{(z - \phi^{-1})^j} dz.$$

Expand the contour to a large circle. The residue of the function under the integral at  $z = -1$  is equal to  $(-1)^j \phi^{-2j}$ . Make a substitution  $z = \frac{1}{w} - 1$ . We are left to calculate the residue at  $w = 0$ . This gives, for  $j \geq 1$ ,

$$a_j = (2\phi - 1) \cdot (1 - (-1)^j \phi^{-2j}) \sim 2\phi - 1, \text{ as } j \rightarrow \infty.$$

We will need the following crucial result.

**Proposition 2.** *Let  $\ell \in \mathbb{N}$ . The  $j$ th  $g$ -coefficient of the function*

$$\frac{(z + 1 - \phi^{-1})^{\ell-1}}{(z + 1 + \phi)^{\ell+1}}$$

*is given by*

$$K(j, \ell) = 5 \cdot 2^{j-\ell-2} \cdot \phi^{-\ell-j} \cdot P_{j-1}^{(\ell-j, 1)}(3/2) = \frac{\phi^{-\ell-j}}{2^{j+1}\pi i} \oint \frac{(w-1)^{\ell-1}(w+1)^j}{(w-3/2)^j} dw; \quad (6)$$

*the contour winds 3/2 in the positive direction. Moreover, we have the symmetry*

$$\ell K(j, \ell) = j K(\ell, j), \quad j, \ell \geq 1.$$

*These coefficients are positive, and  $\sum_{j=1}^{\infty} K(j, \ell) \equiv 1$  for every  $\ell \in \mathbb{N}$ . Further, for fixed  $\ell$ ,  $K(j, \ell)$  achieves its maximum at  $j = \ell$ , and*

$$K(\ell, \ell) \sim \frac{5^{1/4}}{2\sqrt{\pi}\sqrt{\ell}}.$$

*Further,  $K(j, \ell)$  is of fast decay when  $|j - \ell|$  increases.<sup>1</sup>*

*Proof.* By the above remark,

$$K(j, \ell) = \frac{2\phi - 1}{2\pi i} \oint_{\mathcal{C}} \frac{(z + \phi)^j}{(z - \phi^{-1})^j} \cdot \frac{(z + 1 - \phi^{-1})^{\ell-1}}{(z + 1 + \phi)^{\ell+1}} dz.$$

here  $\mathcal{C}$  is a small circle around  $\phi^{-1}$  in the positive direction. Let  $a, b, c, d$  be distinct real numbers. We will generally explore the integral

$$\mathcal{I} = \oint \frac{(z + a)^j (z + b)^{\ell-1}}{(z + c)^j (z + d)^{\ell-1}} \frac{dz}{(z + d)^2};$$

here the contour winds  $-c$  in the positive direction. Let

$$\frac{z + b}{z + d} = w \Rightarrow \frac{z + a}{z + c} = \frac{w(d - a) + (a - b)}{w(d - c) + (c - b)} = \frac{p}{r} \cdot \frac{w + q}{w + s}.$$

So, after this change, the integral  $\mathcal{I}$  transforms into

$$\mathcal{I} = \frac{p^j}{r^j(d - b)} \oint \frac{w^{\ell-1}(w + q)^j}{(w + s)^j} dw.$$

Further, let us make the change  $w \mapsto (q/2)w - q/2$ . This gives

$$\mathcal{I} = \frac{p^j(q/2)^\ell}{r^j(d - b)} \oint \frac{(w - 1)^{\ell-1}(w + 1)^j}{(w - 1 + 2s/q)^j} dw.$$

In our case,  $a = \phi$ ,  $b = 1 - \phi^{-1}$ ,  $c = -\phi^{-1}$ ,  $d = 1 + \phi$ . So,

$$p = d - a = 1, \quad r = d - c = 2\phi, \quad q = \frac{a - b}{d - a} = 2\phi^{-1}, \quad s = \frac{c - b}{d - c} = -\frac{1}{2\phi}, \quad -1 + 2s/q = -\frac{3}{2}.$$

So,

$$\mathcal{I} = \frac{\phi^{-\ell-j}}{2^j(2\phi - 1)} \oint \frac{(w - 1)^{\ell-1}(w + 1)^j}{(w - 3/2)^j} dw.$$

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<sup>1</sup>The needed technical details about  $K(j, \ell)$  will be filled in the next version of this manuscript.

where the contour goes around the point  $w = 3/2$  in the positive direction. This integral can be expressed in terms of Jacobi polynomials, as the definition (4) shows. In our case,  $m = j - 1$ ,  $\alpha = \ell - j$ ,  $\beta = 1$ ,  $x = 3/2$ , and this gives the statement of the proposition.  $\square$

### 3. THE PROOF

**3.1. The main recurrence.** As before, let  $\mathbf{X} \in \mathbb{N}$ . Let us define

$$V^{(0)}(\mathbf{X}) = 1, \quad u^{(0)}(\mathbf{X}, z) = \frac{(z - \phi^{-1})^{\mathbf{X}-1}}{(z + \phi)^{\mathbf{X}+1}},$$

and then the functions  $V^{(v)}(\mathbf{X})$ ,  $u^{(v)}(\mathbf{X}, z)$  recurrently by

$$\sum_{r=0}^v u^{(v-r)}(\mathbf{X}, z) V^{(r)}(\mathbf{X}) = \frac{(-1)^{\mathbf{X}+1} \phi^{2\mathbf{X}}}{(z+1)^2} u^{(v)}\left(\mathbf{X}, \frac{1}{z+1}\right) + \sum_{r=0}^{v-1} \phi^{2\mathbf{X}} u^{(v-r-1)}(\mathbf{X}, z+1) V^{(r)}(\mathbf{X}). \quad (7)$$

This recursion is the core idea of this work: for the unknown function  $u^{(v)}(\mathbf{X}, z)$  we have two terms rather than three, since the last sum is delayed. Also, it turns out that  $V^{(0)}$  is the dominant contributor in the asymptotics of  $\lambda_n$ . This recursion works as follows. If we have already determined  $V^{(r)}$  and  $u^{(r)}$  for  $r \leq v-1$ , the above identity allows to uniquely determine  $g$ -coefficients of  $u^{(v)}$ , except the  $\mathbf{X}$ th coefficient, which can be defined arbitrarily; we set it to be equal to 0. This recursion also yields the unique value for  $V^{(v)}$ , as we will soon see.

To see clearer the main idea of this paper, multiply the identity (7) by  $\phi^{-2\mathbf{X}(v+1)}$ , and sum over  $v \geq 0$ . If we put

$$\Lambda(\mathbf{X}) = \sum_{\ell=0}^{\infty} \phi^{-2\mathbf{X}(\ell+1)} \cdot V^{(\ell)}(\mathbf{X}), \quad U(\mathbf{X}, z) = \sum_{\ell=0}^{\infty} \phi^{-2\mathbf{X}\ell} u^{(\ell)}(\mathbf{X}, z),$$

we obtain the avatar of the initial functional equation

$$\Lambda(\mathbf{X})U(\mathbf{X}, z) = \Lambda(\mathbf{X})U(\mathbf{X}, z+1) + \frac{(-1)^{\mathbf{X}+1}}{(z+1)^2} U\left(\mathbf{X}, \frac{1}{z+1}\right).$$

We will soon show that

$$\lambda_n = (-1)^{n+1} \Lambda(n).$$

**Proposition 3.** *If  $a_j$ ,  $j \geq 1$ , are the  $g$ -coefficients of  $f(z)$ , then the  $g$ -coefficients of*

$$\frac{(-1)^{\mathbf{X}+1} \phi^{2\mathbf{X}}}{(z+1)^2} f\left(\frac{1}{z+1}\right)$$

*are given by  $b_j = a_j(-1)^{\mathbf{X}+j} \phi^{2\mathbf{X}-2j}$ . The  $g$ -coefficients of  $f(z+1)$  are given by  $c_j = \sum_{i=1}^{\infty} a_i K(j, i)$ .*

Let the  $g$ -coefficients of  $u^{(v)}(\mathbf{X}, z)$  be given by  $q_j^{(v)}$ ,  $j \geq 1$  (we omit indication of the dependency on  $\mathbf{X}$ ). Now, let us compare the  $j$ th  $g$ -coefficient of (7). We obtain:

$$\sum_{r=0}^v q_j^{(v-r)} V^{(r)} = q_j^{(v)} (-1)^{\mathbf{X}+j} \phi^{2\mathbf{X}-2j} + \phi^{2\mathbf{X}} \sum_{r=0}^{v-1} \sum_{i=1}^{\infty} q_i^{(v-r-1)} K(j, i) V^{(r)}. \quad (8)$$

For example, when  $v = 0$ , this reads as

$$q_j^{(0)} V^{(0)} = q_j^{(0)} (-1)^{\mathbf{X}+j} \phi^{2\mathbf{X}-2j},$$

and we readily obtain that only  $q_{\mathbf{X}}^{(0)}$  is non-zero; in our case, it is equal to 1. When  $v = 1$ , (8) reads as

$$q_j^{(1)} + q_j^{(0)}V^{(1)} = q_j^{(1)}(-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j} + \phi^{2\mathbf{X}}K(j, \mathbf{X}). \quad (9)$$

When  $j = \mathbf{X}$ , we obtain

$$V^{(1)} = \phi^{2\mathbf{X}}K(\mathbf{X}, \mathbf{X}).$$

(Note that this will be the second largest contributor to the value of  $\lambda_n$ , and it is equal to  $\phi^{-2n}K(n, n)$ , which is of size  $\frac{\phi^{-2n}}{\sqrt{n}}$  - this guarantees the success of our approach!). When  $j \neq \mathbf{X}$ , (9) gives

$$q_j^{(1)} = \frac{\phi^{2\mathbf{X}}K(j, \mathbf{X})}{1 - (-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j}}.$$

As we now see,  $q_{\mathbf{X}}^{(1)}$  can be defined arbitrarily. Indeed, choosing another value for  $q_{\mathbf{X}}^{(\ell)}$ ,  $\ell \geq 1$ , leads to a function  $\tilde{U}(\mathbf{X}, z)$  which is different from  $U(\mathbf{X}, z)$  by a constant factor  $\sum_{\ell \geq 0} q_{\mathbf{X}}^{(\ell)}\phi^{-2\mathbf{X}\ell}$ , provided the last series is absolutely convergent. We therefore always choose  $q_{\mathbf{X}}^{(\ell)} = 0$  for  $\ell \geq 1$ .

As a next step, let us read the recurrence (8) in case  $v = 2$ . This gives:

$$q_j^{(2)} + q_j^{(1)}V^{(1)} + q_j^{(0)}V^{(2)} = q_j^{(2)}(-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j} + \phi^{2\mathbf{X}}K(j, \mathbf{X})V^{(1)} + \phi^{2\mathbf{X}}\sum_{i=1}^{\infty} q_i^{(1)}K(j, i).$$

When  $j = \mathbf{X}$ , this yields

$$V^{(2)} = \phi^{4\mathbf{X}}K^2(\mathbf{X}, \mathbf{X}) + \phi^{4\mathbf{X}}\sum_{i \neq \mathbf{X}} \frac{K(\mathbf{X}, i)K(i, \mathbf{X})}{1 - (-1)^{i+\mathbf{X}}\phi^{2\mathbf{X}-2i}}.$$

Choosing  $j \neq \mathbf{X}$  gives the value for  $q_j^{(2)}$ :

$$\begin{aligned} q_j^{(2)} &= \phi^{4\mathbf{X}} \frac{K(j, \mathbf{X})K(\mathbf{X}, \mathbf{X})}{(1 - (-1)^{j+\mathbf{X}}\phi^{2j-2\mathbf{X}})(1 - (-1)^{j+\mathbf{X}}\phi^{2\mathbf{X}-2j})} \\ &+ \phi^{4\mathbf{X}} \sum_{i \neq \mathbf{X}} \frac{K(j, i)K(i, \mathbf{X})}{(1 - (-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j})(1 - (-1)^{\mathbf{X}+i}\phi^{2\mathbf{X}-2i})}. \end{aligned}$$

The recursion works the same way for  $v \geq 3$ .

**3.2. Asymptotics and convergence.** <sup>2</sup> We write  $f(\mathbf{X}) \cong g(\mathbf{X})$ , if

$$|f(\mathbf{X}) - g(\mathbf{X})| < \frac{C}{\mathbf{X}}$$

for some universal constant  $C$ . Let us take a look at (8) and consider this as equality in the sense of  $\cong$ . Thus, we care only about terms which are of size  $\frac{1}{\sqrt{\mathbf{X}}}$ . By induction, we obtain

$$q_j^{(v)} + q_j^{(0)}V^{(v)} \cong q_j^{(v)}(-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j} + \phi^{2\mathbf{X}}\sum_{i=1}^{\infty} q_i^{(v-1)}K(j, i) + \phi^{2\mathbf{X}}K(j, \mathbf{X})V^{(v-1)}. \quad (10)$$

As we see from Proposition 2, for  $v \geq 2$  the last term drops down, too. When  $j = \mathbf{X}$ , this then gives

$$V^{(v)} \cong \phi^{2\mathbf{X}}\sum_{i=1}^{\infty} q_i^{(v-1)}K(\mathbf{X}, i), \quad v \geq 2.$$

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<sup>2</sup>The needed technical details for this subsection - about the asymptotics of  $K(i, \ell)$  for example - will be filled in the next version of this manuscript.



Next, (10) for  $j \neq \mathbf{X}$  gives

$$q_j^{(v)} \cong \frac{\phi^{2\mathbf{X}}}{1 - (-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j}} \sum_{i \neq \mathbf{X}} q_i^{(v-1)} K(j, i), \quad v \geq 2.$$

These equalities and the asymptotic properties of the coefficients  $K(i, \ell)$  (Proposition 2) imply that

$$|V^{(v)}(\mathbf{X})| < \phi^{2\mathbf{X}v} \cdot \frac{C}{v^2 \sqrt{\mathbf{X}}}, \quad |q_j^{(v)}(\mathbf{X})| < \phi^{2\mathbf{X}v} \cdot \frac{C}{v^2 \sqrt{\mathbf{X}}}$$

for a certain universal  $C$ .

So, we are only left with the following question: how we can be sure that  $(-1)^{n+1}\Lambda(n)$  gives *all* eigenvalues of  $\mathcal{L}$ , that there are no sporadic ones? The answer is provided by the trace formulas, and for this purpose we introduce a generalized operator.

**3.3. Generalized operator.** Let  $\omega \in \mathbb{C}$ ,  $|\omega| \leq 1$ . Now, let us consider the following linear operator  $f : \mathbf{V} \mapsto \mathbf{V}$ , defined by

$$\mathcal{L}_\omega[f(t)](z) = \sum_{m=1}^{\infty} \frac{\omega^{m-1}}{(z+m)^2} f\left(\frac{1}{z+m}\right).$$

This operator is a generalization of (1):  $\mathcal{L}_1[f(t)](z) = \mathcal{L}[f(t)](z)$ . If  $G(\omega, z)$  is the eigenfunction of this operator with the eigenvalue  $\lambda(\omega)$ , then it satisfies the regularity condition (2), and the functional equation

$$\lambda(\omega)G(\omega, z) = \omega\lambda(\omega)G(\omega, z+1) + \frac{1}{(z+1)^2} \cdot G\left(\omega, \frac{1}{z+1}\right), \quad z \in \mathbb{C} \setminus (-\infty, -1].$$

Multiply now the recurrence (7) by  $\omega^v$ . We see that, if we put

$$\Lambda(\omega, \mathbf{X}) = \sum_{\ell=0}^{\infty} \phi^{-2\mathbf{X}(\ell+1)} \cdot \omega^\ell \cdot V^{(\ell)}(\mathbf{X}),$$

then  $\lambda_n(\omega) = (-1)^{n+1}\Lambda(\omega, n)$  is the eigenvalue of  $\mathcal{L}_\omega$  with the eigenfunction  $U(n, \omega, z)$ , where

$$U(\mathbf{X}, \omega, z) = \sum_{\ell=0}^{\infty} \phi^{-2\mathbf{X}\ell} \cdot \omega^\ell \cdot u^{(\ell)}(\mathbf{X}, z).$$

Similarly like  $\mathcal{L}_1$ , for fixed  $\omega$ ,  $|\omega| \leq 1$ , the operator  $\mathcal{L}_\omega$  is of trace class and is nuclear of order zero [16]. Thus,

$$\mathrm{Tr}(\mathcal{L}_\omega) = \sum_{m=1}^{\infty} \frac{\omega^{m-1}}{\xi_m^{-2} + 1}, \quad \mathrm{Tr}(\mathcal{L}_\omega^2) = \sum_{i,j=1}^{\infty} \frac{\omega^{i+j-2}}{(\xi_{i,j}\xi_{j,i})^{-2} - 1}.$$

Now, we know that the eigenvalues of  $\mathcal{L}_\omega$  are analytic functions of  $\omega$  for  $|\omega| < 1$ , and they are real for real  $\omega$ . Let the set of eigenvalues be the union of  $\{\lambda_n(\omega), n \in \mathbb{N}\}$ , and  $\{\sigma_i(\omega), i \in \mathcal{I}\}$ , where  $\mathcal{I}$  is a finite or a countable set. We have:

$$\sum_{n=1}^{\infty} \lambda_n^2(\omega) + \sum_{i \in \mathcal{I}} \sigma_i^2(\omega) \equiv \mathrm{Tr}(\mathcal{L}_\omega^2), \quad |\omega| < 1.$$

In particular, for  $\omega = 0$ , this gives

$$\sum_{i \in \mathcal{I}} \sigma_i^2(0) = \frac{1}{\phi^4 - 1} - \sum_{n=1}^{\infty} \lambda_n^2(0) = 0.$$

Thus, since  $\sigma_i(0) \in \mathbb{R}$  are eigenvalues of  $\mathcal{L}_0$ , this implies  $\mathcal{I} = \emptyset$ . Also, comparing the coefficients at the powers of  $\omega$  in the trace formula, we obtain

$$\sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-2\ell n} V^{(\ell-1)}(n) = \frac{1}{\xi_{\ell}^{-2} + 1}, \quad \ell \geq 1.$$

For  $W_v(\mathbf{X}) = V^{(v)}(\mathbf{X})\phi^{-2\mathbf{X}v}$  this gives the penultimate identity of the Theorem 2. Also, if we compare the coefficient at  $\omega^{\ell-2}$  at the trace of the operator  $\mathcal{L}_{\omega}^2$ , we obtain

$$\sum_{i+j=\ell} \sum_{n=1}^{\infty} \phi^{-2\ell n} V^{(i-1)}(n) V^{(j-1)}(n) = \sum_{i+j=\ell} \frac{1}{(\xi_{i,j}\xi_{j,i})^{-2} - 1}, \quad \ell \geq 1.$$

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## REFERENCES

- [1] G. ALKAUSKAS, The Minkowski question mark function: explicit series for the dyadic period function and moments, *Math. Comp.* **79** (269) (2010), 383–418.
- [2] G. ALKAUSKAS, Recursive construction of a series converging to the eigenvalues of the Gauss-Kuzmin-Wirsing operator, [arXiv:1004.1783](https://arxiv.org/abs/1004.1783).
- [3] K.I. BABENKO, A problem by Gauss, *Dokl. Akad. Nauk SSSR* **238** (5) (1978), 1021–1204; English translation: *Soviet Math. Dokl.* **19** (1) (1978), no. 1, 136–140.
- [4] K.I. BABENKO, S.P. YUR'EV, On a problem of Gauss, *Selecta Math. Soviet.* **2** (1982), 331–378.
- [5] K. BRIGGS (2003), A precise computation of the Gauss-Kuzmin-Wirsing constant, <http://keithbriggs.info/documents/wirsing.pdf>.
- [6] H. DAUDÉ, PH. FLAJOLET, B. VALLÉE, An average-case analysis of the Gaussian algorithm for lattice reduction, *Combin. Probab. Comput.* **6** (4) (1997), 397–433.
- [7] S. R. FINCH, *Mathematical constants*, Encyclopedia of Mathematics and its Applications, 94. Cambridge University Press, Cambridge (2003).
- [8] PH. FLAJOLET, B. VALLÉE (1995), On the Gauss-Kuzmin-Wirsing constant, <http://algo.inria.fr/flajolet/Publications/gauss-kuzmin.ps>
- [9] PH. FLAJOLET, B. VALLÉE, Continued fraction algorithms, functional operators, and structure constants, *Theoret. Comput. Sci.* **194** (1-2) (1998), 1–34.
- [10] PH. FLAJOLET, B. VALLÉE, Continued fractions, comparison algorithms, and fine structure constants, *Constructive, experimental, and nonlinear analysis (Limoges, 1999)*, 53–82, CMS Conf. Proc., 27, Amer. Math. Soc., Providence, RI, 2000.
- [11] D. HENSLEY, *Continued fractions*, World Scientific Publishing Co. Pte. Ltd. (2006).
- [12] A. YA. KHINCHIN, *Continued fractions*, The University of Chicago Press, 1964.
- [13] D. E. KNUTH, *The art of computer programming*, 2nd ed., vol 2: Seminumerical algorithms, Addison-Wesley, 1981.
- [14] J. LEWIS, D. ZAGIER, Period functions and the Selberg zeta function for the modular group, *The mathematical beauty of physics (Saclay, 1996)*, 83–97, Adv. Ser. Math. Phys., 24, World Sci. Publ., River Edge, NJ, 1997.
- [15] A. J. MACLEOD, High-accuracy numerical values in the Gauss-Kuz'min continued fraction problem, *Computers Mathemat. Applic.* **26** (3) (1993), 37–44.

- 
- [16] D. MAYER, On a  $\zeta$  function related to the continued fraction transformation. *Bull. Soc. Math. France* **104** (2) (1976), 195–203.
  - [17] D. MAYER, G. ROEPSTORFF, On the relaxation time of Gauss’s continued-fraction map. I. The Hilbert space approach (Koopmanism), *J. Statist. Phys.* **47** (1-2) (1987), 149–171.
  - [18] D. MAYER, G. ROEPSTORFF, On the relaxation time of Gauss’ continued-fraction map. II. The Banach space approach (transfer operator method)., *J. Statist. Phys.* **50** (1-2) (1988), 331–344.
  - [19] D. MAYER, The thermodynamic formalism approach to Selberg’s zeta function for  $\mathrm{PSL}(2, \mathbb{Z})$ , *Bull. Amer. Math. Soc. (N.S.)* **25** (1) (1991), 55–60.
  - [20] D. MAYER, On the thermodynamic formalism for the Gauss map, *Comm. Math. Phys.* **130** (2) (1990), 311–333.
  - [21] D. MAYER, Continued fractions and related transformations. *Ergodic theory, symbolic dynamics, and hyperbolic spaces* (Trieste, 1989), Oxford Sci. Publ., Oxford Univ. Press, New York, 1991 (175–222).
  - [22] P. SEBAH, Computing eigenvalues of Wirsing’s operator (2012); <http://mif.vu.lt/~alkauskas/MP3/eigen-sebah.txt>.
  - [23] G. SZEGŐ, *Orthogonal polynomials*. Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I. (1975).
  - [24] E. WIRSING, On the theorem of Gauss-Kusmin-Lévy and a Frobenius-type theorem for function spaces, *Acta Arith.* **24** (1973/74), 507–528.
  - [25] D. ZAGIER (2001), New points of view on the Selberg zeta function, Proceedings of the Japanese-German Seminar “Explicit Structures of Modular Forms and Zeta Functions”, Ryushi-do (2002) <http://people.mpim-bonn.mpg.de/zagier/files/tex/NewPointsSelbergZeta/fulltext.pdf>.

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